Natural Codes as foundation in hierarchical labeling and extended hexadecimals for variable-length bit strings

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ABSTRACT

We propose a positional notation for bit strings (binary sequences) with an arbitrary amount of bits, which preserves hierarchy and is compatible with the ordinary hexadecimal numeric representation. When the elements of a nested set are labeled with natural numbers, the hierarchical structure is not preserved; but when the numerical labels are replaced by variable-length bit strings, distinguishing leading zeros (00 and 0 as distinct entities), it is possible to express labels with hierarchical syntax, preserving the original nested structure, and classical hexadecimals as subset. We present a formal definition of this type of numerical labeling system, calling it Natural Codes. We also show that it can be transformed into human-readable encodings (positional notations such as base4, base8 or base16), the base notation extensions base4h, base8h and base16h for hierarchical representation. The reference-model adopted is the finite Cantor set with hierarchy degree *k* — strictly, the set C_k is the "hierarchical tree of the Cantor set", which is isomorphic to a complete binary tree —, proposing a simple labeling process, mapping elements of C_k into elements of the Natural Codes. The proposal of the base notation extensions introduces a complement to ordinary base syntax, where the last digit can use, when necessary, a complementary alphabet to represent partial values (with one bit less than ordinary digits). The hierarchical representation is a superset, differenciating from ordinary base notation by this last digit, that is a non-hierarchical member of the representation (named *nhDigit*). We also offer algorithms for implementing the conversion of the base extensions, and we discuss optimizations.

key-words: human-readable encoding, hierarchical labeling, leading zeros, base conversion.

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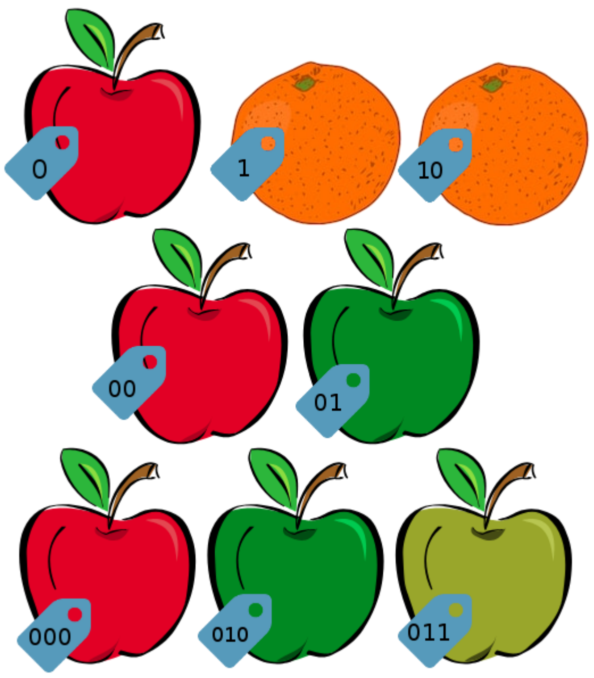
SUMMARY

Introduction



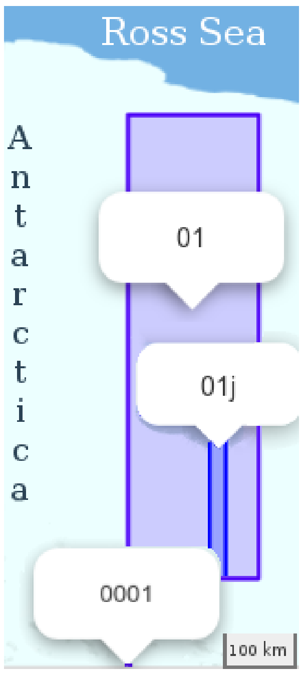
We use natural numbers (ℕ) for counting and labeling, but sometimes, to represent hashes, geocodes, or hierarchical indexes, we need a kind of "number" where **0 is not equal to 00**, while preserving properties of natural numbers like order (e.g. 011>010) and the freedom to translate its positional notation to some other base (e.g. binary to hexadecimal).

In the set of apples illustrated, each apple was identified by a variable-length bit string, so the set *A* of the identifiers is
  A = { `0`, `00`, `000`, `01`, `010`, `011` }.
It's easy to separate apples from other fruits, because all apple labels starts with "`0`". It's also easy to select green apples, because they have the "`01`" prefix, *G* = { `01`, `010`, `011` } ⊂ *A*.
This characteristic of differentiation through **prefix preservation in subsets of labels** can be important, and it is only possible to express with bit strings — as we will show, strictly speaking, for infinite sets, it is impossible with only ℕ.

Each element of *A* can also be interpreted as a binary number.
For example, the decimal value of binary `01` is 1, or with base-subscript notation, $[01]_2=[1]_{10}$. Also $[010]_2=[2]_{10}$ and $[011]_2=[3]_{10}$, but there is some loss of information when adopting equivalence in $[0]_2 = [00]_2 = [000]_2 = [0]_{10}$.
There is a loss of uniqueness of the labels of red apples, all of which have value zero but 1, 2 or 3 bits of size.

A solution to avoid this information loss is to transform each bit string into a pair of decimal numbers (*size,value*):
  A' = { (1,0), (2,0), (3,0), (2,1), (3,2), (3,3) }.
Interpreting some samples: the element `0` of *A* was transformed into (1,0) of *A'*, it has a size of 1 bit and numeric value zero, $[0]_{10}=[0]_2$; the element `011` was transformed into (3,3), it has a size of 3 and value $[3]_{10}=[11]_2=[011]_2$.

Neither the bit strings of *A*, nor the equivalent pairs of *A'* have a specific mathematical designation, but there are many applications that make use of such labels, so we have named them
Natural Codes[8] (the name "Sized Naturals" was also used in first versions).



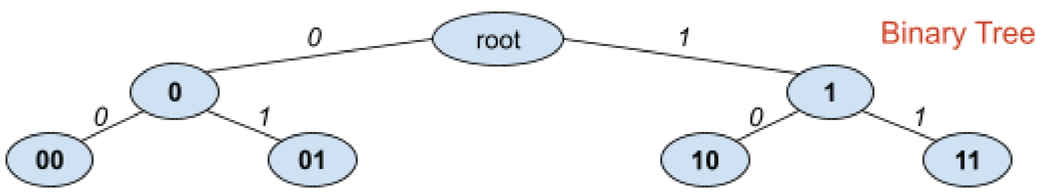
Examples of real life encoding systems that exhibit such labeling characteristics:

- Two checksums (e.g. CRC32 digests) with different lengths are distinct, we can't eliminate leading zeros.
  The CRC32 human-readable standard representation is hexadecimal preserving leading zeros. E.g. `000fa339` is not equal to `fa339`.

- Two Geohashes with different lengths are distinct, e.g. `01` is a cell identifier of a geographic location little below Ross Sea with ~115000 km², and `0001` is a cell far below, with 3.5 km². There is also a cell `01j` inside `01` with ~2700 km². All the illustrated cells have with the same first-digit prefix, they are contained into the bigger `0` cell.
  The Geohash encoding system uses base32 as human-readable standard representation, but internally the encode function uses the bit string.

[8] Adopting the spelling of a proper name, despite the analogy with the standard "natural numbers" (a common noun), precisely because it is not a standard.

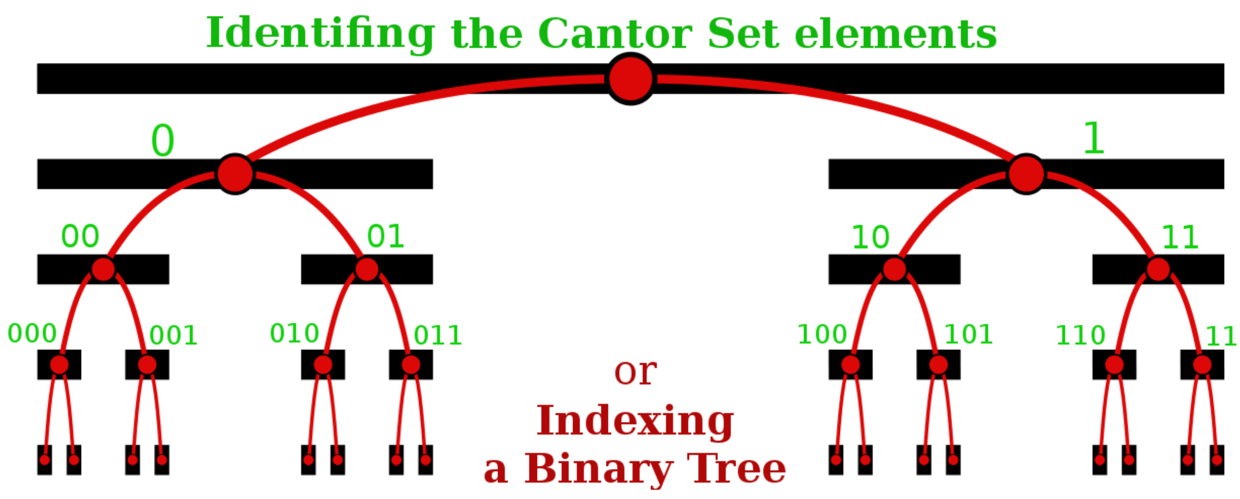
- Indexing a binary tree where the left edges are labeled 0 and the right edges are labeled 1: a node is labeled by the concatenated string of branch labels from the root to the node. Example: to reach node 01 from root, take the path of edges labeled 0 and 1.

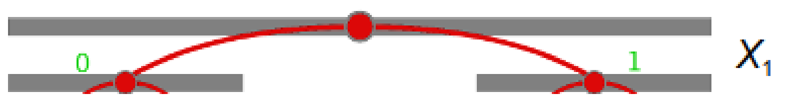


The purpose of this article is to define Natural Codes and a hexadecimal representation for them.

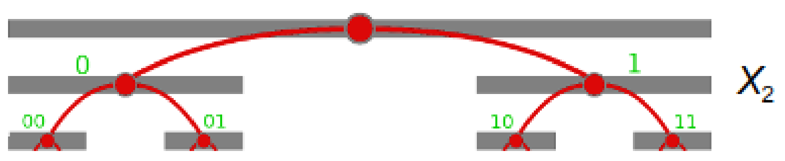
Indexing hierarchical items with bit strings

In Mathematics, the Cantor set is used as reference model to express concepts like self-similarity and subdivision rule. In Computation, the equivalent reference model is the complete binary tree. We can use either one here as reference for indexing and hierarchical labeling. Let's use the Cantor Set.

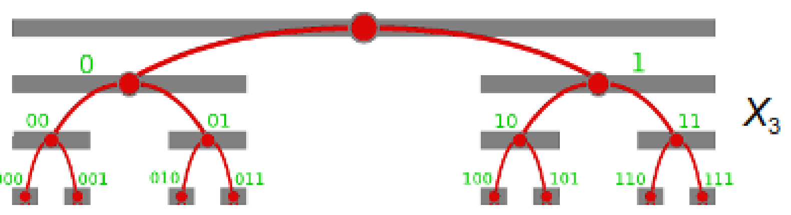




Each element of the "tree of the Cantor set" (red), a "Cantor subset" (black), can be uniquely labeled by a bit string, therefore an identifier (ID), similar to the labelling of the set of apples.



Each ID retains information of its left (0) or right (1) positions, as well as its vertical position in the hierarchy of elements — each ID has a prefix that is the ID of its parent.
Example: the parent of `011` is `01`.



Adopting the convention that **set X_k** is the set of identifiers of the Cantor subsets limited to **k-bits**.
X_1 = {0, 1};
X_2 = {0, 00, 01, 1, 10, 11};
X_3 = {0, 00, 000, 001, 01, 010, 011, 1, ..., 111};
X_k = ...;
X_8 = {0, 00, 000, 000, ..., 11111110, 11111111}.
In usual applications, the set X_k is the domain of a set of identifiers (or indexes), and *k* is finite.

Example: the set X_3 is the domain of the illustrated set A of apples, $A \subset X_3$.

There is an intuitive recursive construction rule (illustrated) for each new X_k after X_1

$X_k = P_k \cup X_{k-1}$

where ***P*_k is the set of all of 2^k numbers expressed as fixed-length (*k*) bit strings** — and opportune to remember that the elements of P_k can be mapped (casted) to the Naturals of the range 0 to *k-1* by its binary representation. Example:

$X_2 = P_2 \cup X_1 = \{00, 01, 10, 11\} \cup \{0, 1\}$.

The number of elements, $|X_k|$, after $|X_1|=2$, is the recursion $|X_k| = |X_{k-1}| + 2^k$.

Examples, for *k* ranging from 2 to 8:

$|X_2| = 2+4=6$; $|X_3| = 6+8=14$; $|X_4| = 14+16=30$; ...; $|X_8| = 254+256=510$.

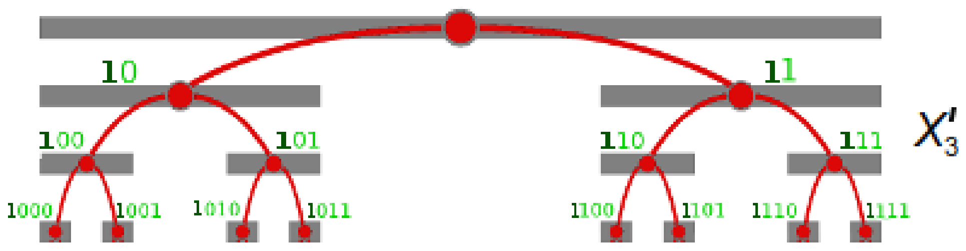
By induction we see that $|X_k| = 2^{k+1} - 2$. As suggested by this power *k*+1, is possible to map any Natural Code (*l*,*n*) to a number $n+2^{l+1}$, because it preserves *n* and the leading zeros information.

The **hierarchy of the elements of P_k** is visible in the illustration above: the last line, that is the subset P_k of new members, has elements *x* formed by the concatenation (operator $\oplus$) of prefix *p* and suffix *s*, $x=p\oplus s$, where *p* is the parent identifier and *s* is the left or right label. Example: the parent `01` of `010=01⊕0` and of `011=01⊕1`.

The hidden bit implementation strategy

It's possible to "protect leading zeros" and represent a finite Natural Code by a natural number. Any Natural Code (*l*,*n*) can be mapped to a number $n+2^l$ without loss of the leading zeros.

The illustration shows the Natural codes of *k*=3, transformed in a set *X'* of Naturals, where the first bit is the value 2^l added to *n* of each original (*l*,*n*) element of X_3.



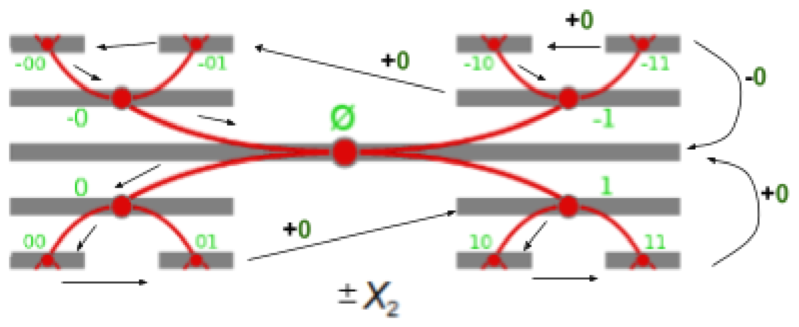
This strategy can be used to simplify the internal representation in algorithms and, in some circumstances, as a mathematical alternative to the bit string or ordered pair representations.

In order to "externalize" this internal numeric representation as bit string we must remove the first bit, so we can say that the first bit is "hidden" from the outside of the storage and its algorithms.

In Set Theory (as foundational system), is important to remember that a set of "all bit strings" is isomorphic to a set of Naturals by this transformation, only when the set is finite (or countably infinite) and all its elements are finite. Otherwise, by Cantor's theorem, there cannot exist a one-to-one function from the set of infinite-length bit strings to $\mathbb{N}$.

For finite sets, for example in a computer with 16 bits fixed-length unsigned integer, the conversion is a right-copy. For example for bit string "010" and left-side-hidden-bit concatenation (results "1010"), the internal integer representation will be "0000000000001010". Operations like "+1" make sense on same set *P*, will be "1011" in this example. And there is a second way to implement it in fixed-length: as left-copy. In the same 16 bits example the left-copy and right-hidden-bit, for "010", will result in "0101000000000000". It loss the consistent integer number representation and its operations (the "+1" results in "0101000000000001"). But it preserves, on bitwise operations, the hierarchical prefix: it is easy and has good performance on hierarchical analysis, see section Algorithms for details.

Empty and inverse elements



A finite set, to be a first-class citizen in Mathematics, must be the generator of a *group*. Suppose the illustrated ***G* = (*X*₂ ∪ {∅}, +)**.

The introduction of the simbol "∅" as the neutral element of cyclic addition is useful for bit string representation, because "empty string" is a valid value. The unit is the element "0", to follow the "+0" lexicographical order in this representation system.

Any "*x*+0" of X_k , when *length(x)<k*, is a concatenation, like in an unary numeral system. For example "1+0=10". Otherwise "*x*+0" is a Turing Machine operation, analysing the last digit of *x*: when "0" replace it by "1", else cut "1"s and replace residual "0" by "1". When cut them all the result is ∅, as "11+0=∅" in the X_2 example.
The illustrated group *G* is isomorthic to the quotient group of seven elements, ℤ/7ℤ. It is possible to see also any bit string as a *p-adic number* with p=2, so it is possible to conceive *G* as a *ring of p-adic integers*.

Problem on positional notations

It is not possible, with usual non-binary positional notations like hexadecimal, to represent all Natural Codes. Even the most simple, base4, as illustrated by question marks below.

Table 1

(size,value)	BitString	Base4	Base16
(1,0)	0	**?**	**?**
(2,0)	00	0	**?**
(3,0)	000	**?**	**?**
(4,0)	0000	00	0
(5,0)	00000	**?**	**?**
(6,0)	000000	000	**?**
(7,0)	0000000	**?**	**?**
(8,0)	00000000	0000	00
(8,1)	00000001	0001	01
(7,1)	0000001	**?**	**?**
(8,2)	00000010	0002	02
(8,3)	00000011	0003	03
(6,1)	000001	001	**?**
(7,2)	0000010	**?**	**?**
(8,4)	00000100	0010	04
(8,5)	00000101	0011	05
...	...	...	...

The table above shows that the pairs (*l*,*n*) of the first column, titled (*size*,*value*), can always be represented by a bit string of the second column, and vice-versa, but not always in base4 or base16.

A bit string *x* can be converted to a base *b* only when the number of bits *x_l* of the bit string is a multiple of *dpd=ceil*(*log*$_2$(*b*)), i. e. the number of bits per digit of the base *b*. Expressing in terms of remainder (modulo operation), it is only possible when ***x_l* % dpd = 0**. Examples: in Table-1 the column *base4* doesn't have question marks, "?", when the size is 2, 4, 6 or any other even size. That's because when *b*=4 we need $\log_2(4)=2$ bits/digits. The column *base16* doesn't have "?" when the size is 4, 8 or any multiple of 4, because when *b*=16 we need $\log_2(16)=4$ bits.

Testing a naive solution and requirements to the optimal

Seeing how the simplest strategy doesn't work. A common solution to preserve leading zeros is to use an external symbol instead of a leading zero. For example the RFC 4648, section 3.2, suggests:

(...) *the use of padding ("=") in base-encoded data* (...).

Let's use the letter "`Z`" as external symbol. Using the illustrated set *A* of apples to exemplify:
A = { `0`, `00`, `000`, `01`, `010`, `011` } = { `0`, `Z0`, `ZZ0`, `Z1`, `Z10`, `Z11` }
Can we preserve this external symbol when we translate the set to base4? The natural tanslations are:
`Z0⇒0; Z1⇒1; 10⇒2; 11⇒3; ZZ⇒Z.` For example "`ZZZZ10`" to "`ZZ2`" and "`ZZ11`" to "`Z3`".
But what about "`0`", "`1`", "`ZZ0`" or "`ZZ1`"? To translate the bit string representation to its analog base4 string we need to add more external symbols (beyond "`Z`"), suppose `A`, `B`, `C`, `D` and `E`:
`0⇒A; Z0⇒C; ZZ0⇒ZA; 01⇒1; Z10⇒DA; Z11⇒DE`. This translation results in
A' = { `A`, `C`, `ZA`, `1`, `DA`, `DE` }. No hierarchy is perceived, and only element "`1`" ressambles to ordinary base4.

Conclusion: extending base4 alphabet with symbols "`Z`", "`A`", ..., "`E`" does not seem useful. When translating leading zeros from base2, we lost hierarchy visualization. The demand for a lot of new external symbols is also a problem, since human-readability requires something simple that resembles base 4 in most of its representation.

The ideal (optimal) representation is a superset for ordinary base 4 and include a minimal alphabet for the exotic elements: we will show in the next sections that ideal representation exists, by simply adding a new last digit when it's needed. In the example of set *A*, we will show that the solution results in *A'* = {`G`, `0`, `0G`, `1`, `1G`, `1Q`}, where 83% of the elements (5 in 6) resembles ordinary base 4: or translation is exact ("`0`" from `00⇒0` and "`1`" from `01⇒1`), or preserves hierarchy (commom prefixes has been preserved at "`0G`", "`1G`" and "`1Q`").

Objectives

The aim of this document is to define Natural Codes, review its foundations and dedicate to the public domain the algorithms of its representation in the extended positional notations:

- the **base4h** representation for bit strings of arbitrary length, that is the "ordinary base4 whith leading zeros" when the length of the bit string is even, and an extension of base4 when the length is odd.
- the **base8h** representation for bit strings of arbitrary length, that is the ordinary base8 when the length is a multiple of 3, and an extension of base8 when it's not.
- the **base16h** representation for bit strings of arbitrary length, that is the ordinary base16 (hexadecimal) when the length is a multiple of 4, and an extension of base16 when it's not.

In the context of software development and programming languages, it's expected to promote *data types* (of hashes, geocodes, hierarchical indexes, etc.) to first-class citizens based on Natural Codes.

License

Formal definition of Natural Codes

The fundamentals of Set Theory are described in Halmos (1960), and the convention for the set of natural numbers (e.g. adopting 0 as element of ℕ) are reinforced in ISO 80000-2:2009. The classic reference-models for the *bit string* concept are the "strip of tape of bits" of Turing (1937) and the binary message of Shannon (1948). "The set of the **Natural Codes**" is a particular name for "the set of all bit strings", and the following is a formal definition oriented towards the introduced labeling applications. The Natural Codes are also indifferent in regards to the element's representation, as bit strings, as ordered pairs of numbers or other.

We can transform all elements of ℕ into bit strings, through the element's binary representation (base2), forming the set *L* of labels of ℕ. Still, *L* is a subset of Natural Codes because bit strings can use leading zeros, representing distinct elements.

The bit-length of a bit string is its number of bits. Otherwise, the standard way to express the "size" of an element of ℕ is through the number of digits in its base2 representation, which is known as the bit-length of the number. To avoid confusion we adopt the "minimum Bit-Length" (minBL) function:

$$minBL(n) = \begin{cases} \lceil log_2(n) + 1 \rceil & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

All Natural Codes **with bit-length *k*** can be expressed as a set $\boldsymbol{P_k}$

$$P_k = \{\forall x = (k, n) \mid n \in \mathbb{N} \ \wedge \ minBL(n) \leq k\}$$

The finite set $\boldsymbol{X_k}$ **of all Natural Codes** with **maximum bit-length *k*** can be defined by recursion:

$$X_k = \begin{cases} P_k \cup X_{k-1} & \text{if } k > 1 \\ P_1 & \text{if } k = 1. \end{cases}$$

Table-1 shows an example, a sample of X_8 set. Because of finite size of the real-world fractal structures, the set X_k of labels must be mapped into finite Cantor set, as described by Merlo et al. (2003). In this context, the elements of a set P_k can be used as ordered labels of the Cantor bars of level *k*.

> Terminological note. All the elements of X_k can be mapped to a full hierarchical structure of level *k*, $C_k = \{X_1, X_2, \ldots, X_k\}$, the "hierarchical collection of Cantor sets" (see next section). Using analog construction rules, Merlo et al. named it also as "hierarchical tree of the Cantor set".

Any element of X_k can be expressed as bit string by the function *toBitString*(*l*,*n*), that is the binary representation of *n* with padding zeros to *l*. They are semantically equivalent: the bit string and the ordered pair representations.

The **hierarchy** expressed by the recursion is embedded in both element representation options:

- as a bit string *x* of length *l*, it has a prefix of length *l*-1 that is the bit string representation of its parent.
- as an ordered pair, *x*=(*l*,*n*), the parent is the pair (*l*-1,*m*) where ***m* = *floor*(*n* / 2)**.

When the hierarchy doesn't need to be explicit, we can use a simplified definition of X_k

$$X_k = \{\forall x = (l, n) \mid l, n \in \mathbb{N} \ \wedge \ minBL(n) \leq l \leq k\}$$

The functions *minBL*(*n*) and *toBitString*(*l*,*n*) are also expressed in Javascript at appendix.

All above definitions also apply to the term “Sized Integers” used in some implementations. As in usual generalizations from $\mathbb{N}$ to $\mathbb{Z}$, can be accomplished with some control and carrying the sign.

Hierarchy as nested set collection

A *nested set* is a set containing a chain of subsets, forming a hierarchical structure. In Set Theory it is related to partial order, and nested sets are used as reference for any *hierarchy* or class *inheritance* definitions.

Let *B* be a non-empty set and *C* be a collection of subsets of *B*. Then *C* is a *nested set collection* if:

$$B \in C$$
$$\forall H, K \in C \ : \ H \cap K \neq \emptyset \ \Rightarrow \ H \subset K \ \vee \ K \subset H$$

So we can suppose that the collection *C* that represent the k-Natural Codes hierarchy is defined by
$C_k = \{X_1, X_2, ..., X_k\} \ = \ \{ P_1, \ P_1 \cup P_2, ..., P_1 \cup P_2 \cup \ ... \ \cup P_k\}$
where the union operations are demonstrating that the second condition is satisfied. So always is true that
$X_1 \subset X_2 \subset X_3 \subset ... \subset \ X_k$
and that the collection C_k is a partial order for “⊂”, so a **strict containment order**. In other words, we can say that X_3 is the parent of X_2 that is the parent of X_1. The index *i* convention ensures that $i<a$ imply $X_i \subset X_a$.

The **hierarchy of the elements**, most important and commented before, can be expressed by the relations between prefixes of the elements of a set X_i and its parent X_{i-1}. Every element *x* of X_i with more than one bit,

- the bit string representation is a concatenation of a prefix *p* and a suffix *s*, $x=p \oplus s$, where $p \in X_{i-1}$.
- in the pair representation $x=(l,n)$ and $p=(l\text{-}1, floor(n / 2)) \in X_{i-1}$.

Hidden bit representation

The hidden bit strategy is consistent (see also practical implementations). There is a bijective relation
$f\colon X_k \rightarrow H_k \subset \mathbb{N}$ and its inverse f^{-1} that maps any Natural Code into a natural number,

$$f(l,n) = n+2^l \qquad\qquad f^{-1}(m) = [minBL(m)\text{-}1 \ , \ m\text{-}2^{minBL(m)}].$$

The function *f* adds one bit to the value, and f^{-1} removes it. When Natural Code is expressed by bit strings *i* of the set $\boldsymbol{P_k}$ the function produces the set $\boldsymbol{P'_k}$ of numbers, and can be expressed as

$$f(i) = \text{cast\_to\_integer(bitwise\_or}(i,2\text{\^{}}k)\text{)}.$$

The construction of H_k follows the recurrence rule, $\boldsymbol{H_k} \ = \boldsymbol{P'_k} \cup \ \boldsymbol{H_{k-1}}$.

In programming languages we can promote an entity to first-class citizen when it supports all the operations generally available to other primitive entities. That is the case, the elements of H_k are primitive (e.g. Bigint data type), and any integer function can be used — e.g. integer comparizon to check equals, or specialized `m_compare_lexOrder(a,b)` of the appendix to sort, or any bitwise function to extract prefixes. The main application is to use it as a 64 bits primary key in a database.

Bit string and base2h notation

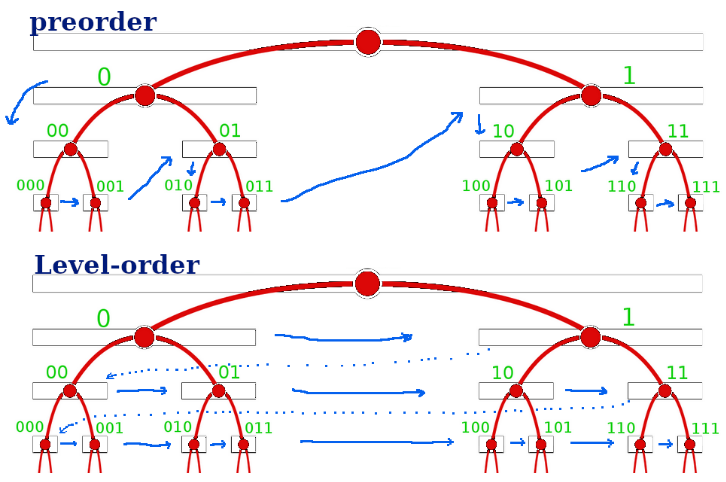
In this article, to represent *bit string* values without ambiguity, we adopted the base-subscript notation with “2h” standing “hierarchical base2”. Examples: $[011]_{2h}$, $[000]_{2h} \neq [00]_{2h}$.

Similarly a sequence can be expressed in brackets. For instance, the elements of X_2 in lexicographic order can be expressed by a sequence: $S_2=[0, 00, 01, 1, 10, 11]_{2h}$.

Strictly speaking, it is not valid to compare a Natural with a SizedNatural, except when casting datatypes. Example: toBase2($[00]_{2h}$) = $[0]_2$. For internal representations (hidden bit) it is necessary to assign equivalence, for example to say that $[00]_{2h}$ is internally mapped to $[100]_2$.

Lexicographic and numeric orderings

An ordering is a method for sorting elements, and it can be based on a *comparison algorithm* or conversion to a natural number. Mathematically the Natural Code set X_k is an well-order set when every non-empty subset of X_k has a least element (by the chosen *comparison*). For practical applications there are two main alternatives:



- binary **lexicographic order**: the bit “0” is the least in the set {0,1}, that is the alphabet of the bit string. The same result in a binary tree, the Cantor Set labeled by Natural Codes, listing codes during traverse, is known as *preorder* (illustrated).
  Note: in computers the direct bit string comparison is the faster algorithm (scan and compare each bit), but when using pair representation, other algorithms can be used to avoid casting (see appendix).

- numeric **level order**: is the “natural order” of the numeric (right-copy) hidden bit representation.
  The *level order* in a binary tree (illustrated) is known by the same name.
  Example using X_2 in a bit string representation but numeric level order sequence,
  $S'_2=[0, 1, 00, 01, 10, 11]_{2h}$.
  When using the pair representation, (l,n), the algorithm to compare a and b is:
  if $(a.l \neq b.l)$ them $a.l<b.l$ else $a.n \leq b.n$.

For hidden-bit internal representation is possible to implement lexicographic order without bit string convertions. The algorithm is based into a function that count the leading zeros, *prefix_length*(x), and can be expressed by a non-recursive algorithm. The *sort method* calls a *compare(a,b)* function that returns a boolean, false if *a* is “less” than *b*, null if they are “equal”, and true if the *a* is “greater”. In the appendix we demonstrate that the function exists (see *m_compare_lexOrder*) and eventually can be optimized in low-level implementations (in C or assembler languages), to be faster than the “cast to bit string”.

Positional representation of Natural Codes

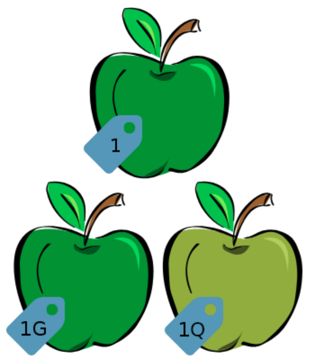
Natural numbers can be expressed with positional notation, using the rule of "remove leading zeros". The rule is used in any base (radix) representation. The Natural Code's representation is like “natural numbers *without the rule of remove leading zeros*”, and not affects prefix hierarchy in any of its valid base representation. The solution to accomplish forbidden conversions will be explained in the next subsections: as illustrated in Table-2, is to extend base4 and base16 alphabets, adding a last digit when forbidden, a digit with new symbols (examples in bold below).

Table 2

(size,value)	BitString	Base4h	Base16h	(size,value)	BitString	Base4h	Base16h
(1,0)	0	**G**	**G**	(1,1)	1	**Q**	**Q**
(2,0)	00	0	**H**	(2,2)	10	2	**R**
(3,0)	000	0**G**	**J**	(3,4)	100	2**G**	**S**
(4,0)	0000	00	0	(4,8)	1000	20	8
(4,1)	0001	01	1	(4,9)	1001	21	9
(3,1)	001	0**Q**	**K**	(3,5)	101	2**Q**	**T**
(4,2)	0010	02	2	(4,10)	1010	22	a
(4,3)	0011	03	3	(4,11)	1011	23	b
(2,1)	01	1	**M**	(2,3)	11	3	**V**
(3,2)	010	1**G**	**N**	(3,6)	110	3**G**	**Z**
(4,4)	0100	10	4	(4,12)	1100	30	c
(4,5)	0101	11	5	(4,13)	1101	31	d
(3,3)	011	1**Q**	**P**	(3,7)	111	3**Q**	**Y**
(4,6)	0110	12	6	(4,14)	1110	32	e
(4,7)	0111	13	7	(4,15)	1111	33	f
...	...	...	...	...	...	...	...
(8,29)	...	0131	1d	(10,117)	...	01311	1d**H**
(9,58)	...	0131**G**	1d**G**	(9,59)	...	0131**H**	1d**Q**
(10,116)	...	01310	1d**H**	(10,118)	...	01312	1d**R**

The column Base4h on Table 2 shows that there is prefix preservation, ensuring that hierarchical subsets can be obtained by simple syntax inspection. The elements of the illustrated subset of the green apples, *G*, subset of the set *A* of apples, can be selected by the prefix “1”.

A = {“G“, “0“, “0G“, “1“, “1G“, “1Q“}   *G* = {“1“, “1G“, “1Q“} ⊂ *A*

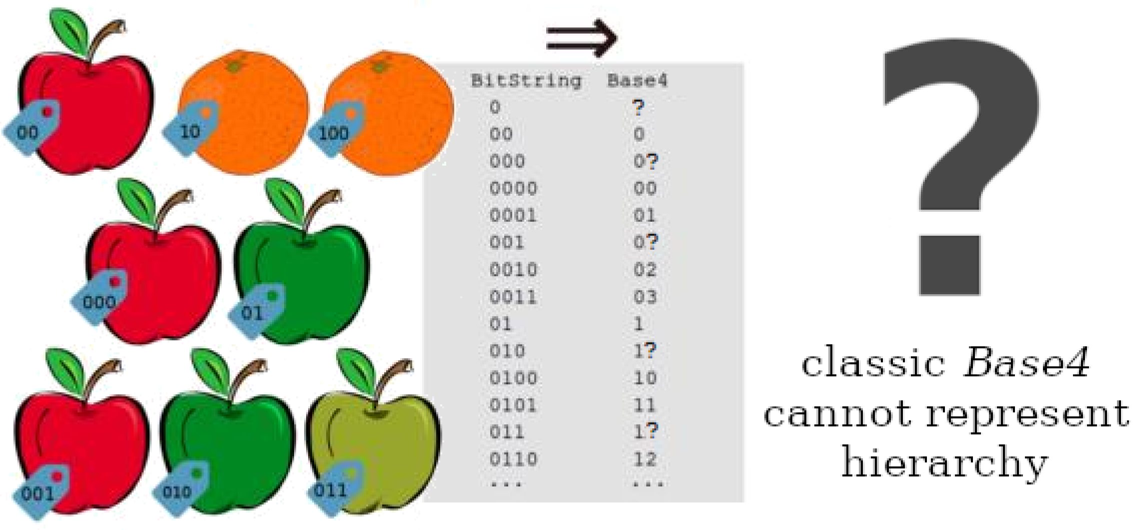


Bit string or base2h

The *bit string* representation is the simplest and the **canonic** one.
The Natural Code *bit string* representation is the base2h: the ordinary base2 augmented with the "use leading zeros" rule (`00` and `0` are distinct elements).

Base4h

How to convert one-digit *bit strings* `0` and `1` to base4? Or the *bit strings* like `000`?
The translations $[00]_{2h}=[0]_4$ and $[01]_{2h}=[1]_4$ make sense for *bit strings*, but there are no standard rule or tradition for translate 1 bit, 3 bits, 5 bits etc. *bit strings*. $[0]_{2h}=[?]_4$; $[1]_{2h}=[?]_4$; $[000]_{2h}=[?]_4$.



BitString	Base4
0	?
00	0
000	0?
0000	00
0001	01
001	0?
0010	02
0011	03
01	1
010	1?
0100	10
0101	11
011	1?
0110	12
...	...

The solution is to use a fake digit that represent these values. To avoid confusion with hexadecimal letters we can use `G` to represent `0` and `Q` to represent `1`. It is named "**n**on-**h**ierarchical digit" (**nhDigit**), because, despite accommodating hierarchical representation, it is not a member of the hierarchy. The "**base4** extended for **h**ierarchy" was shortened to "base**4h**". Table-2 is illustrating base4h representation of all elements of the X_4 set.

Base4h numbers are strings with usual base4 pattern and the *nhDigit* as optional suffix. This syntax rule, to recognize arbitrary base4h codes, can be expressed by a regular expression:

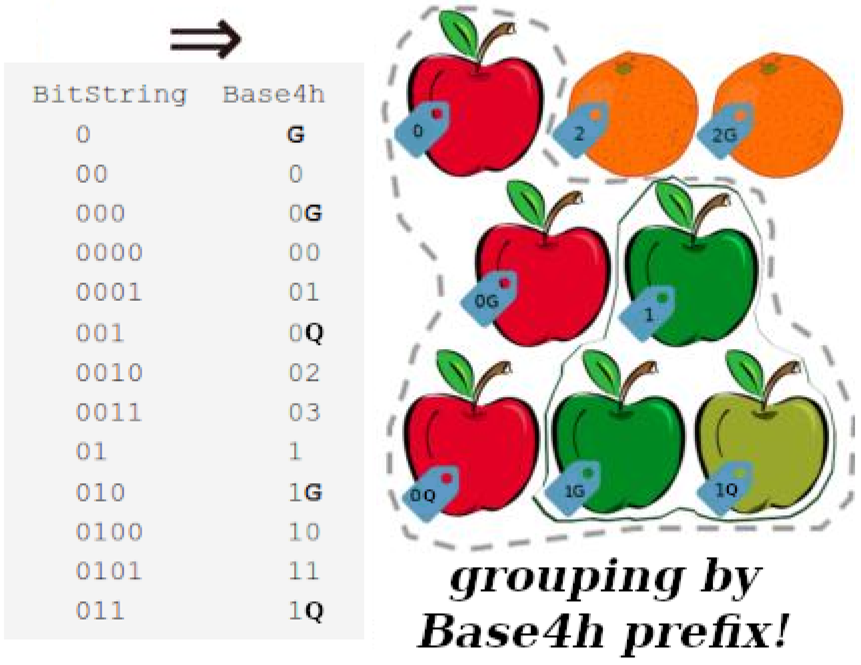
```
/^([0123]*)([GQ]?)$/
```

The inverse, to translate from bit string with *b* bits, when *b* is even we can use ordinary base4 conversion, and when *b* is odd, concatenate the *nhDigit*. Splitting (e.g. with Javascript) the binary value as prefix and suffix parts,

```
let part = bitString.match(/^((?:[01]{2,2})*)([01]*)$/)
```

the prefix (part[0]) will be translated to usual base4 number, and the suffix (part[1]), when it exists (a remaining last bit) will be translated to *nhDigit* by this JSON map: `{"0":"G","1":"Q"}`.

Example: to convert `001010010` into base4h, split into parts, part[0]=`00101001` of 2-bits blocks from begin, that will result in "0221", and part[1]=`0`, of remaining bit, resulting in "G". Concatenating part results, "0221G".



BitString	Base4h
0	**G**
00	0
000	0**G**
0000	00
0001	01
001	0**Q**
0010	02
0011	03
01	1
010	1**G**
0100	10
0101	11
011	1**Q**

Base16h

This encoding extension for base16 (RFC 4648, sec 8) was inspired in the base4h encode. It uses the same *nhDigit* concept: a complementary syntax to ordinary base representation where the last digit can use an alternative alphabet to represent partial values (with less bits) of ordinary digits.

We can use hexadecimal representation for any integer, but when controlling the bit-length can use only base16-compatible lengths: 4 bits, 8 bits, 12 bits, ... multiples of 4.
So, how to transform into base16 the bit strings like `0`, `1`, `00`, `01`, `10`, ... ?

The solution is to extend a hexadecimal representation, in a similar way to the previous one used for base4h: the last digit as a fake-digit that can represent all these incompatible values — so using the *nhDigit* values `G` and Q for 1-bit values, and including more values for 2 bits (4 values) and 3 bits (8 values). The total is 2+4+8=14 values, they can be represented by the letters `G` to `T` or any other set of 14 letters — **table 3** shows our choices.

The set was optimized excluding vowels (`I`,`O`,`U`) and symbols that may be easily confused with each other (as `L`), and excluding `X` because is used as hexadecimal convertion prefix (e.g. `x0123`).

The name of this new representation is **base16h**, because it is the "ordinary *base16* extended for *h*ierarchy":

`/^([0-9a-f]*)([GHJKMNP-TVZY])?$/`

The inverse, to translate from bit string with *b* bits, there are *b*%4 last bits to be translated to a *nhDigit*. Splitting (e.g. with Javascript) the value as prefix and suffix parts,

```
let part = bitString.match(/^((?:[01]{4,4})*)([01]*)$/)
```

the prefix (*part[0]*) will be translated to usual hexadecimal number, and the suffix (*part[1]*), when exists (with 1, 2 or 3 last bits), will be translated by this "last bits to *nhDigit*" JSON map:

```
{ "0":"G", "00":"H", "000":"J", "001":"K", "01":"M", "010":"N", "011":"P",
  "1":"Q", "10":"R", "100":"S", "101":"T", "11":"V", "110":"Z", "111":"Y"
}
```

Example: to convert `0010100101` into base16h, split into part[0]=`00101001` of 4-bits blocks from begin, and part[1]=`01`, of remaining bits. Convert part[0] into ordinary hexadecimal (`00101001` is "29"), and part[1] by the JSON table above (`01` is "M"), so it results in "29M".

Base8h

This encoding is less usual, but use the same pattern and implementation tham base16h.

String-detection pattern: `/^([0-7]*)([GQHMRV])?$/`
Split into parts: `bitString.match(/^((?:[01]{3,3})*)([01]*)$/)`
JSON map for *nhDigit*: `{ "0":"G", "00":"H", "01":"M", "1":"Q", "10":"R", "11":"V" }`

Example: to convert bit string `0010100101` into base8h, split into part[0]=`001010010` of 3-bits blocks from begin, and part[1]=`1`, of remaining bits. Convert part[0] into ordinary octal (`001010010` is "122"), and part[1] by the JSON table above (`1` is "Q"), so it results in "122Q".

The Natural Code algebra

The base4h, base8h and base16h representations are compact expressions of the bit string representation, and all can be converted to each other without restrictions. Examples of functions and operations:

- Formal group algebra: as demonstred before, the set of Natural Codes can be generalized with the "+" operator and "∅" as identity element, forming a cyclic group.
- Complementar operations: prefix remove, prefix concatenation and other practical operations are valid in bit string and base-h representations — provided that a suffix of sufficient length for the base-h last digit.
- Positional representation: any code x of the set of all Natural Codes can be represented as bit string or a "base *n*h" with *n* in {2,4,8,16}. There is a function **encode_baseh(*x*,*n*)** that returns an human-readable expression *s* for *x*, using an alphabet of *n* symbols, and its inverse **decode_baseh(*s*,*n*)** that returns *x*. Also, there is a kind of encode/decode algebra for *cast* operations over internal 64 bits representation.
- Order by different criteria. The bit string ascending order is defulat, but descending, or level-order, or binary-tree traversals orders, etc. are also valid.

Base-h order and collation

To simplify base-h, we reuse alphabets:
alphabet(*base4h*) ⊂ *alphabet*(*base8h*) ⊂ *alphabet*(*base16h*).

Another important decision, to keep the standard numerical representations as a subset of base-h:
base4 ⊂ *base4h*; *base8* ⊂ *base8h*; *base16* ⊂ *base16h*.

We pay a high price for the last decision when we are ordering a list of codes: "base16 numeric order" and "base16h bitstring order" are interleaved. The aim is the bit string lexicographical order, as in the table beside, where the colored cells show the interleaving problem.

The best that we can do, for human-reader, is to preserve order in the set of hexadecimal digits, and the order in the set *nhDigits*. The complete (union of sets) order must use, for example, a translation in the code's last digit.

Assuming a column *x* of natural codes in base16h, in a SQL database. The "ORDER BY x" will fail. It needs a character translation, here expressed in PostgreSQL:

```
SELECT x FROM t ORDER BY traslate(
  x,
  'GHJ01K23MN45R67QRS89TabVZcdYef',
  '0123456789ABCDEFGHIJKLMNOPQRST'
) -- or only last-digit translation
```

To be friendly, avoiding translations, the base-h codes must be represented by *text* with special collation. Supposing that base-h will be a standard, it is natural to propose a collantion for it. Imagine defining a column as base16h representation:

```
CREATE TABLE t (x text COLLATE "num_b16h");
```

Table 3

bitstring	base4h	base8h	base16h	Order	OrderedChar
0	G	G	**G**	1	0
00	0	H	**H**	2	1
000	0G	0	**J**	3	2
0000	00	0G	**0**	4	3
0001	01	0Q	**1**	5	4
001	0Q	1	**K**	6	5
0010	02	1G	**2**	7	6
0011	03	1Q	**3**	8	7
01	1	M	**M**	9	8
010	1G	2	**N**	10	9
0100	10	2G	**4**	11	A
0101	11	2Q	**5**	12	B
011	1Q	3	**P**	13	C
0110	12	3G	**6**	14	D
0111	13	3Q	**7**	15	E
1	Q	Q	**Q**	16	F
10	2	R	**R**	17	G
100	2G	4	**S**	18	H
1000	20	4G	**8**	19	I
1001	21	4Q	**9**	20	J
101	2Q	5	**T**	21	K
1010	22	5G	**a**	22	L
1011	23	5Q	**b**	23	M
11	3	V	**V**	24	N
110	3G	6	**Z**	25	O
1100	30	6G	**c**	26	P
1101	31	6Q	**d**	27	Q
111	3Q	7	**Y**	28	R
1110	32	7G	**e**	29	S
1111	33	7Q	**f**	30	T
Extra-alphabet:	GQ	+HMRV	+JKNPSTZY		

Base 32rh and others

The number of human-readable new digits in the nhDigit alphabet (and the ability of remember its order) is limited. In this article we recommend limiting it to the base 16h.

Any other base *b* representaion preserving hierarchy is possible by leading zeros and usual restriction to convert bit strings with restricted sizes, the multiples of $\log_2(b)$ bits, illustrated before in Table-1. To avoid confusion with the ordinary base conversion, we adopt the terminology “base *b* with **r**estricted **h**iearchy”. For exemple “**base 32 with restricted hiearchy**”, or short “**base32rh**” — it is not “base32” because use leading zeros and it is not “base32h” because it can't be used perfectly in an algebra of Natural Codes.

For exemple Geohash codes “6g” and “6g55” are base32rh representations, and can be converted to hexadecimal (base16) representation, but “6” and “6g5” can’t. Codes with an odd number of digits are not convertible (only to base16h instead base16).

NOTE. The same definition is not so useful but is also valid also for base4rh, base8rh and base16rh. That said, it should be remembered that, strictly speaking, ordinary base conversion does not apply to bit strings due to the loss of information. We must use only “h” and “rh” bases for bit string representation.

Algorithms

There are many ways to store internally a Natural Code of our formal definition. The choice is a matter of context (e.g. database or interface), simplicity and language. Main options for internal representation:

s	bit string. Two ways: ○ bits: real buffer of bits. Example: *bit varying* datatype of PostgreSQL. Best case. ○ characters: using the ASCII characters "0" and "1", in a text string. Worst case.
(*l*,*n*)	length and value. Small integer for *l* and "big integer" for *n*. Used in formal definition.
(*p*,*n*)	padded zeros and value. Small integer for *p* and "big integer" for *n*. Not used.
m	only a value, using the hidden-bit strategy. The same "big integer" *n*, but *m* is unsigned, and internally uses more one bit to protect leading zeros.

Any of these options can be optimized when values are limited to, e.g. 32 or 64 bits.

All algorithms and conventions presented in this article were tested using Javascript, using Web context constraints and performance evaluations. The SizedNatural concept was adapted to the Javascript primitive type BigInt. We also implemented SizedNatural concept in PostgreSQL database, using the hidden bit strategy. In a future release we will develop C++ code to run as database library or WebAssembly Javascript object.

Bit strings into integers

Finite natural numbers are represented in computers by *unsigned integers* of fixed length, typically 64 bits (8 bytes), and eventually by a variable-length in arbitrary-precision arithmetic frameworks. For simplicity, many databases and programming languages do not offer unsigned, only ***signed*** *integers*. Therefore, **to use the hidden bit strategy** is necessary to adapt the bit string to regular integers.

There are no IEEE standards, but the "standard" internal representation for regular integers is the following:

- positive values, like unsigned integers, are represented from left to right, ending with the unit (less significative bit) in the right.
- the signal is the first (leftmost) bit. Zero is positive and 1 is negative.
- negative values are represented as "two's complement". When signal changes, all numeric representation also changes.

The hidden bit representation with regular integers depends on the implementation context. The **conversion from bit string to integer**, in order to be optimized, it must be specified as a whole. The first decision is the copy, related with the two different internal representations, the ***left_copy*** and and ***right_copy***:

original	left_copy	right_copy
011100	[**0**_011100_**1**000000000000000000000000]$_2$ = [956301312]$_{10}$	[**0**000000000000000000000000**1**_011100_]$_2$ = [92]$_{10}$

In PostgreSQL it can be performed by this query:

```
SELECT *, left_copy::int, right_copy::int FROM (
  SELECT x AS original, (b'0' || x || b'1')::bit(32) AS left_copy,
     overlay( b'0'::bit(32) PLACING (b'01'||x) FROM 31-length(x) ) AS right_copy
  FROM (select b'011100') t(x) ) t2; -- any bitstring with less than 31 bits
```

In order to preserve original variable-size information into the fixed-size representation: the left copy must be protected on the right, and the right copy on the left. For ordering and for human intuition the best is to use *right_copy*. Some systems, like S2 Geometry, use the *left_copy*, and has some rationale for it: it is easy and fast to get a fix-lenght hierarchical prefix, the most frequently used operation. Anyway *left_copy* doesn't follow the lexicographical order.

On arbitrary precision integers, a candidate to variable-length representation, is rare to implement it directly as BitArray, they are optimized for CPU word (e.g. of 8, 32 or 64 bits) and big-endian/little-endian representations.

The main context-algorithm choices are summarized in the table below. Notation: ***x*** is the bit string, ***s*** is the signal '0' bit, ***h*** the '1' left-mark for hidden bit *right_copy*, ***k*** the '1' right-mark for hidden bit *left_copy*. The symbol "$\oplus$" is the concatenation operator.

Context	**Loss of bits** (n:description)	**Algorithm's label and spec. summary**
Arbitrary precision, unsigned int	1: *h*	**A1**: Concatenation $h \oplus x$ and cast.
Arbitrary precision, signed int	2: *s* and *h*	**A2**: $s \oplus h \oplus x$ and cast.
Fixed length, unsigned int8	1: *h*	**A3R**: $h \oplus x$ and cast to right.
Fixed length, unsigned int8	1: *k*	**A3L**: $x \oplus k$ and cast to left.
Fixed length, int8	2: *s* and *h*	**A4R**: $s \oplus h \oplus x$ and cast to right.
Fixed length, int8	2: *s* and *k*	**A4L**: $s \oplus x \oplus k$ and cast to left.

NOTES: 1. the optimized implementation is not the specification. In C-language for example we can use the *union* datatype and copy directly the bit string into adequate position. 2. the (signed) integer algorithms not use the signal bit to avoid complexity. 3. In this article we are limited to discussion of the tested implementations, for Javascript (algorithm A2) and PostgreSQL (algorithm A4L).

Examples:

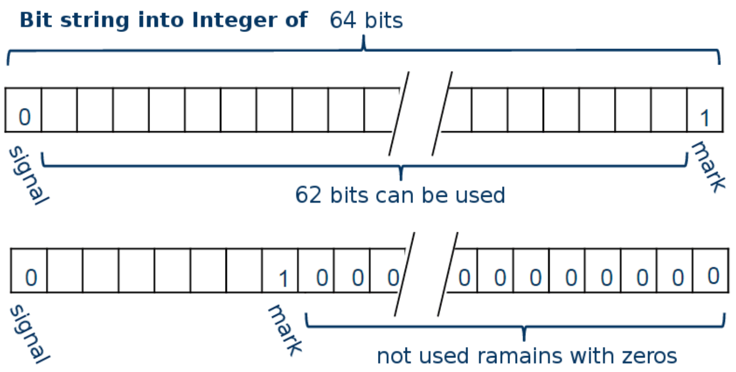
algor.	**enc([*0*]$_{2h}$) =**	**enc([*00*]$_{2h}$) =**	**enc([*1*]$_{2h}$) =**	**enc([*001*]$_{2h}$) =**
A1	[**1***0*]$_{2h}$ = [2]$_{10}$	[**1***00*]$_{2h}$ = [4]$_{10}$	[**1***1*]$_{2h}$ = [3]$_{10}$	[**1***001*]$_{2h}$ = [9]$_{10}$
A2	[**01***0*]$_{2h}$ = [+2]$_{10}$	[**01***00*]$_{2h}$ = [+4]$_{10}$	[**01***1*]$_{2h}$ = [+3]$_{10}$	[**01***001*]$_{2h}$ = [+9]$_{10}$
A3R	[0…0**1***0*]$_{2h}$ = [2]$_{10}$	[0…0**1***00*]$_{2h}$ = [4]$_{10}$	[0…0**1***1*]$_{2h}$ = [2]$_{10}$	[0…0**1***001*]$_{2h}$ = [9]$_{10}$
A3L	[*0***1**0…0]$_{2h}$ = [4611686018427387904]$_{10}$	[*00***1**0…0]$_{2h}$ = [2305843009213693952]$_{10}$	[*1***1**0…0]$_{2h}$ = [1383…3712]$_{10}$	[*001***1**0…0]$_{2h}$ = [3458…0928]$_{10}$
A4R	[**0**0…0**1***0*]$_{2h}$ = [+2]$_{10}$	[**0**0…0**1***00*]$_{2h}$ = [+4]$_{10}$	[**0**0…0**1***1*]$_{2h}$ = [+3]$_{10}$	[**0**0…0**1***001*]$_{2h}$ = [+9]$_{10}$
A4L	[*0***1**0…0]$_{2h}$ = [2305843009213693952]$_{10}$	[*00***1**0…0]$_{2h}$ = [1152921504606846976]$_{10}$	[*1***1**0…0]$_{2h}$ = [6917…1856]$_{10}$	[*001***1**0…0]$_{2h}$ = [1729…0464]$_{10}$

The internal representation of algorithm A4L is illustrated below.

Signed implementation

Illustrating the *left_copy* case.

Javascript only has the option to represent larger natural numbers through the native Bigint datatype. The best algorithm for this context is algorithm A2. Two bits are lost, with 62 bits remaining to be used as Natural Code. The appendix contains Javascript implementations.



Int8 implementation

PostgreSQL offers direct bit string representation (named *varbit*) and *int8* (named *bigint*). The conversion from varbit to bigint is easy with both algorithms, A4R and A4L. Using the same examples, to reproduce the four columns of the table, the SQL statement is:

```
 SELECT 'A4R' alg, b'10'::bigint x0, b'100'::bigint x1, b'11'::bigint x2, b'1001'::bigint x3
  UNION
SELECT 'A4L', b'0010000000000000000000000000000000000000000000000000000000000000'::bigint,
 b'0001000000000000000000000000000000000000000000000000000000000000'::bigint,
 b'0110000000000000000000000000000000000000000000000000000000000000'::bigint,
 b'0001100000000000000000000000000000000000000000000000000000000000'::bigint
;
```

The generic functions, taking the original bit string *x* as input, use `varbit_to_bigint(b'01'||x)` for algorithm A4R and `(b'0'||x||b'1')::bit(64)::bigint` for algorithm A4L. Two bits are lost, 62 bits are left to store the Natural Code *x*.

To get back the bit string, it is possible to use substring bit string operations to remove the extra bit with knowledge of its position. In the case of algorithm A4R, the position can be obtained from integer base2-logarithm. In the case of algorithm A4L, the “right-most non-zero bit position” needs a specialized function that is non-optimized, except by implementing with C-language, using the CLZ function.

Comparing Natural Codes

There no constraint in our Natural Code definition about order, any one is valid. We adopted the *bit string lexicographic order* as canonic (see the line sequence at tables 1 and 2), that corresponds, in the Complete Binary Tree models, to the pre-order traversal. See functions `_compare_lexOrder(a,b)` at appendix. It is canonic because in a listing it groups same-prefix items. The most simple, `pair_compare_lexOrder(a,b)` uses `pair_toBitString(l,n)` for basic string comparison. For best performance the ideal internal representation is *m* (hidden bit), that is direct numeric comparison — can be tested with `m_compare_lexOrder(a,b)`.

The library offers also the level-order. The `m_compare_levelOrder(a,b)` function it is an integer comparison. The `pair_compare_levelOrder(a,b)` function is implemented by *l* and *n* numerical comparisons: when a.*l*=b.*l* it compares a.*n* and b.*n*.

Trucating

The most usual is to truncate a prefix, see `_truncate(x,bits)` function at appendix. Sometimes it is useful to split a Natural Code into both a prefix and suffix for a new Natural Code.

Base conversion

Mathematical libraries, like the native Javascript BigInt, have good performance in ordinary base conversion (see *BigInt.toString*(*radix*) method). The most frequently used ones are 4, 8,16, 32 and 64. The alphabet can be controlled by some standardization, see example in the appendix. The base4h and base16h, as shown above, have additional performance cost to split the bit string into prefix (that use fast native conversion of ordinary base) and *nhDigit*, that is a fast key-value conversion.

Tested implementations

To test concepts of the article and algorithms described above, we implemented them in two flavors: Javascript and SQL. It is pending a complementary C++ library to offer optimized versions of critical critical functions, like the integer logarithm and the CLZ funcion, or perhaps also base conversion functions.

All implementations were done with Apache v2.0 license and can be accessed at git.osm.codes.

Javascript SizedBigInt

Sized BigInt's are arbitrary-precision integers adapted to represent Natural Codes. It is a Javascript class, using primitive data type BigInt in internal representation. The class also was extended to represent discrete global grid cell identifiers, like Geohash, and its encoding options.

See also Krauss (2019) with online demonstration of labeling quadrilateral grid cells of Space-filling curves.

SQL NatCod

We also implemented Natural Codes concept in PostgreSQL database, using *bit strings* (the *bit varying* datatype), and optional hidden bit strategy for usual fixed 64 bits integers (the *bigint* datatype of the database). The bigint is useful for indexes and internal operations, and some cases of intermediary data interchange.

Terminology

Base: the web standards, as RFC 4648, use the term "base", but Javascript (ECMA-262) adopted the term "radix" in `parseInt(string,radix)`. The preferred term is *base*.

Base alphabet: is the "encoding alphabet", a set of UTF-8 symbols used as digit values of a specific base.

Base label: each pair (base,alphabet) need a short label. In the SizedBigInt class some labels was defined: base2, base4, base4h, base8, base16, base16h, base32, base32ghs, base32hex, base32pt, base32rfc, base64, base64url, base64rfc. See also the ID column of the *catalog-base.csv* file at class' git repository.

Default alphabet: is the alphabet adopted as standard for a specific base, associated with the label "baseX", for example "base4" is a synonym for "base4js", the ECMA-262 standard for it. See *base label***.**

Padding: SizedBigInt's are numbers where padding zeros make difference (0 is not equal to 00). The RFC 4648 is not for numbers, the convention of pad "=" characters at the end of encoded data was adopted.

Set, **element**, **number**, **natural number** and **Integer** are terms of the Set theory (Halmos 1960), the formal mathematical foundation used here. In implementation context the **Javascript** semantic for integer, class, number, etc. is preferred.

Size and **length**: the term "size" was used in the title of this project, but the usual term for "size of the string" is length and, for binary numbers, bit-length, the preferred term.

Sized BigInt: term used in the first Javascript library for Natural Codes, where the term BigInt is used to express arbitrary-leght integer numbers.

Sized Integer: the formal definition expressed in this article is about natural numbers (positive Integers), but it is easy to generalize, this was only a simplification in order to avoid signal analysis and to reuse implementations.

Unsigned Sized Integer: synonym for Natural Code. In Computation the term “unsigned integer” is preferred in place of “natural number”.

References

Standards:

ECMA-262 v9.0 (2018), “ECMAScript® 2018 Language Specification”.

ISO 80000-2:2009, "Quantities and units—Part 2: Mathematical signs and symbols to be used in the natural sciences and technology", Chapters 5 and 6.

RFC 4648 (2006), "The Base16, Base32, and Base64 Data Encodings".

Other works:

J. Ferreirós (2007), "Labyrinth of Thought, A History of Set Theory and Its Role in Modern Mathematics", Second revised edition. ISBN 978-3-7643-8349-7.

P. Halmos (1960), "Naive Set Theory". ISBN 978-1-61427-131-4.

P. Krauss, E. Bortolini, T. Jean, R. P. Julião (2018), “Geohash Generalizado: uma metodologia para a implementação de funções de hash geradoras de geocódigos hierárquicos base32, Versão 0.4”, urn:doi:10.5281/zenodo.2529558.

P. Krauss (2019), “SFC4Q Space-filling curve valid to index hierarchical grid of aperture-4 quadrilateral cells”, software source-code. urn:doi:10.5281/zenodo.7112018.

P. Krauss, Claiton (2023), “NaturalCodes Library”. https://git.osm.codes/NaturalCodes

D. R. Merlo, J. A. R. Martín-Romo, T. Alieva, M. Calvo (2003), “Fresnel Diffraction by Deterministic Fractal Gratings: An Experimental Study”, urn:doi:10.1134/1.1595227 (open).

C. E. Shannon (1948), “A Mathematical Theory of Communication”. urn:doi:10.1002/j.1538-7305.1948.tb01338.x.

L. A. Steen, J. A. Seebach (1995), “Counterexamples in Topology”. ISBN 978-0-486-68735-3.

A. M. Turing (1937), “On Computable Numbers, with an Application to the Entscheidungsproblem”. urn:doi:10.1112/plms/s2-42.1.230.

Appendix - supplementary material

The lexicographic adition and its cyclic group

As showed in this mathematical proof example, it is long and deserves a separate article for a group theory contextualization of the Natural Codes. The choice of zero have two alternatives and must be also dicussed.

Basic Natural Codes functions

These functions are used only as functional specification and didactic reference, **not** for optimized implementations neither offering pre-conditions or exceptions. All were done in Javascript (ES6+), and at functions, the input parameter “*l*” can be a “*Number(l)*”; input parameter “*n*” must be a “*BigInt(n)*”; input parameter “*x*” must be an array of size, “x[0]” and value, “x[1]”. Name prerfix “pair_” and “m_” refer representation.

Utility functions:

```
function minBL(n){
 // calculates bigInt_log2(n)+1, the minimum bit length of BigInt n.
 for(var count=0; n>BigInt(1); count++)  n = n/BigInt(2)
 return count+1
}
function pair_minBL(x) {  return minBL(x[1])  }
function m_minBL(m) { return minBL(m)-1 }
```

Functions using pairs (*l*,*n*) as internal representation:

```
// javascript Array convention x[l,n]. So x[0] is l and x[1] is n.

function pair_toBitString(x) {
  // transforms BigInt of (l,n) representation into a bit string representation.
  return x[1].toString(2).padStart(x[0],'0')
}

function pair_fromBitString(s) {
  let l = s.length
  let n = BigInt("0b"+s)
  return [l,n]
}

function pair_truncate(x,bits) {
  return pair_fromBitString( pair_toBitString(x).slice(0,bits) );
}

function pair_compare_lexOrder(a,b) {
  // compare in lexical (ASCII) order the two arrays
  let str_b = pair_toBitString(b)
  return pair_toBitString(a).localeCompare(str_b)
  // same as (str_a>str_b)? 1: ( (str_a==str_b)? 0: -1 )
}


function pair_compare_levelOrder(a,b) {
  let bitsDiff = a[0] - b[0]   // compare bitLengths (l)
  if (bitsDiff) return bitsDiff;
  else { // when equal lengths, compare BigInts (n)
    let valDiff = a[1] - b[1]
    return valDiff? ((valDiff>BigInt(0))? 1: -1): 0
  }
}
```

```
function pair_lexOrder_next(x,maxBits=null,cycle=false) {
       // the successor of x in a context of lexicographical order, returning string
       let t = x[0]  // x[0] is the size, x[1] the value
       if (!t) return null;
       if (!maxBits) maxBits=t; else if (t>maxBits) return null;
       let s = pair_toBitString(x)
       if (t<maxBits) return s+'0';
       t--
       if (s[t]=='0') return s.slice(0,t)+'1';
       else return (s==''.padEnd(maxBits,'1'))? (cycle?'0':null): s.slice(1);
}
```

Functions using *m* of the hidden bit strategy as internal representation:

```
function m_toBitString(m) {
  // transforms hidden_bit BigInt representation into a bit_string representation.
  return (m===null)? '': m.toString(2).slice(1)
}

function m_fromBitString(s) {
  return s? BigInt("0b1"+s): null
}

function m_truncate(m,bits) { // can be optimezed
  return m_fromBitString( m_toBitString(m).slice(0,bits) )
}

function m_compare_lexOrder(a,b) {
  // compare two SizedBigInt of m representation
  let dif;
  let bdif = m_minBL(a) - m_minBL(b)
  if (bdif) {
       dif = (bdif>0)
         ? a/BigInt(2**bdif) - b      // normalize a
         : a - b/BigInt(2**(-bdif));  // normalize b
       if (!dif) dif = bdif;    // 0 before 00, 101 before 0101 etc.
  } else
      dif = a - b
  return dif? ((dif>BigInt(0))? 1: -1): 0
}
function m_compare_levelOrder(a,b) {
  let dif = a - b
  return dif? ((dif>BigInt(0))? 1: -1): 0
}
```

Encoding alphabets and conventions

Complete list of standard alphabets for base conversion. Only “power of 2” bases. The identifier is the concatenation of word “base”, the value of the base and the alphabetLabel.

base	alphabet label	id	bits /digit	alphabet (after space *nhDigits*)	Reference standard
2	js*	base2js	1	`01`	ECMA-262
4	js*	base4js	2	`0123`	ECMA-262
4	h	base4h	2	`0123 GQ`	ECMA + nhDigits alphabet
8	js*	base8js	3	`01234567`	ECMA-262
8	h	base8h	3	`01234567 GQ HMRV`	ECMA + nhDigits alphabet
16	js*	base16js	4	`0123456789abcdef`	ECMA-262 and RFC 4648/sec8
16	h	base16h	4	`0123456789abcdef GQ HMRV JKNPSTZY`	ECMA + nhDigits alphabet
32	hex*	base32hex	5	`0123456789abcdefghijklmnopqrstuv`	ECMA-262 and RFC 4648/sec7
32	ghs	base32ghs	5	`0123456789bcdefghjkmnpqrstuvwxyz`	Geohash
32	nvu	base32nvu	5	`0123456789BCDFGHJKLMNPQRSTUVWXYZ`	No-Vowels except U (near non-syllabic)
32	rfc	base32rfc	5	`ABCDEFGHIJKLMNOPQRSTUVWXYZ234567`	RFC 4648/sec6
64	url*	base64url	6	`ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghij klmnopqrstuvwxyz0123456789-_`	RFC 4648/sec5
64	rfc	base64rfc	6	`ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghij klmnopqrstuvwxyz0123456789+/`	RFC 4648/sec4

(*) default base. For example base32 is interpreted by default as base32hex.
All base32 here are using with leading zeros, they are valid instances of “restricted hiearchy”, *base32rh*.